

# ON THE $L^p$ -THEORY OF THE NAVIER-STOKES EQUATIONS ON THREE-DIMENSIONAL BOUNDED LIPSCHITZ DOMAINS

PATRICK TOLKSDORF

**ABSTRACT.** On a bounded Lipschitz domain  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 3$ , we continue the study of Shen [28] and of Kunstmann and Weis [20] of the Stokes operator on  $L_\sigma^p(\Omega)$ . We employ their results in order to determine the domain of the square root of the Stokes operator as the space  $W_{0,\sigma}^{1,p}(\Omega)$  for  $|\frac{1}{p} - \frac{1}{2}| < \frac{1}{d} + \varepsilon$  and some  $\varepsilon > 0$ . This characterization provides gradient estimates as well as  $L^p$ - $L^q$ -mapping properties of the corresponding semigroup. In the three-dimensional case this provides a means to show the existence of solutions to the Navier-Stokes equations in the critical space  $L^\infty(0, \infty; L_\sigma^3(\Omega))$  whenever the initial velocity is small in the  $L^3$ -norm. Finally, we present a different approach to the  $L^p$ -theory of the Navier-Stokes equations by employing the maximal regularity proven by Kunstmann and Weis [20].

## 1. INTRODUCTION

In this article we consider the incompressible Navier-Stokes equations

$$(NSE) \quad \begin{cases} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla \pi = f & \text{in } \Omega, t > 0 \\ \operatorname{div}(u) = 0 & \text{in } \Omega, t > 0 \\ u(0) = a & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, t > 0, \end{cases}$$

on a bounded Lipschitz domain  $\Omega \subset \mathbb{R}^3$ . Here,  $u$  denotes a vector field which corresponds to the velocity of an incompressible fluid that governs  $\Omega$ ,  $\pi$  the pressure inside  $\Omega$ ,  $f$  an external force, and  $a$  the initial velocity. While there exists extensive literature to this equation if  $\Omega$  is smooth, see, e.g., [21, 29, 33], the investigation for bounded Lipschitz domains started fairly recently. For example one of the first existence results of strong solutions was given by Deuring and von Wahl [8] in 1995 and was ultimately improved by Mitrea and Monniaux [23]. In both articles the authors followed the classical approach of Fujita and Kato [10]. Another existence result — proven in a very short and elegant way — was given by Taylor [32]. However, it has to be noted, that all approaches use only  $L^2$ -theory in order to establish the existence theorems.

It is well-known, that also the  $L^p$ -theory is of great interest, for example for uniqueness and regularity questions. However, an  $L^p$ -theory seemed to be out of reach for a long time as it was not even known that the Stokes operator generates a strongly continuous semigroup on  $L_\sigma^p(\Omega)$  for  $p \neq 2$ . Due to the boundedness properties of the Helmholtz projection, see [9], Taylor conjectured in [32], that for each bounded Lipschitz domain, there exists  $\varepsilon > 0$  such that the Stokes operator generates an analytic semigroup on  $L_\sigma^p(\Omega)$  whenever  $3/2 - \varepsilon < p < 3 + \varepsilon$ . Taylor's conjecture was answered to the affirmative by Shen in the seminal paper [28].

To establish an  $L^p$ -theory for the Navier-Stokes equations there are several ways one can take and we will present two of them in this article. The first is the classical approach to obtain

---

*Date:* March 6, 2017.

*2010 Mathematics Subject Classification.* 35Q30, 76D05, 76D07, 76N10.

*Key words and phrases.* Navier-Stokes equations, Strong solutions, Lipschitz domains, Maximal regularity, Gradient estimates.

The author was supported by "Studienstiftung des deutschen Volkes".

mild solutions via an iteration scheme, which was first performed by Giga [15] on bounded and smooth domains and by Kato [18] on the whole space. This approach requires certain  $L^p$ - $L^q$ -mapping properties as well as gradient estimates of the Stokes semigroup. These estimates will be established in this work by using the boundedness of the  $H^\infty$ -calculus of the Stokes operator proven by Kunstmann and Weis [20].

The second approach we present uses the maximal  $L^q$ -regularity of the Stokes operator. This approach was initiated by Solonnikov [30] and largely improved in the subsequent years, see, e.g., [1, 4, 5, 12]. The property that the Stokes operator on  $L_\sigma^p(\Omega)$  has maximal  $L^q$ -regularity was recently proven by Kunstmann and Weis [20] for the same range of numbers  $p$  as the analytic semigroup exists. Independently, this property was proven in the PhD-thesis of the author, see [34].

For a further historical review the reader may consult the introductions in [8, 23, 32].

In the remainder of this introduction, we formulate the main results presented in this article. We refer to Section 2 for the respective notation. Note that the results dealing with the linear theory are formulated in  $\mathbb{R}^d$  with  $d \geq 3$ .

**Theorem 1.1.** *Let  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 3$ , be a bounded Lipschitz domain. Then there exists  $\varepsilon > 0$  depending only on  $d$  and the Lipschitz character of  $\Omega$  such that for all*

$$\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{1}{d} + \varepsilon$$

*the domain of the square root of the Stokes operator  $A_p$  on  $L_\sigma^p(\Omega)$  coincides with  $W_{0,\sigma}^{1,p}(\Omega)$ , i.e.,*

$$\mathcal{D}(A_p^{1/2}) = W_{0,\sigma}^{1,p}(\Omega)$$

*with equivalence of the respective norms.*

The second theorem dealing with the linear theory gives the  $L^p$ - $L^q$ -mapping properties of the Stokes semigroup as well as the gradient estimates.

**Theorem 1.2.** *Let  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 3$ , be a bounded Lipschitz domain. Then there exists  $\varepsilon > 0$  depending only on  $d$  and the Lipschitz character of  $\Omega$  such that for all  $p \leq q$  that satisfy*

$$\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{1}{d} + \varepsilon \quad \text{and} \quad \left| \frac{1}{q} - \frac{1}{2} \right| < \frac{1}{d} + \varepsilon$$

*the Stokes semigroup satisfies the estimates*

$$\|e^{-tA_p} f\|_{L_\sigma^q(\Omega)} \leq C t^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})} \|f\|_{L_\sigma^p(\Omega)} \quad (f \in L_\sigma^p(\Omega))$$

*and*

$$\|\nabla e^{-tA_p} f\|_{L^p(\Omega; \mathbb{C}^{d^2})} \leq C t^{-\frac{1}{2}} \|f\|_{L_\sigma^p(\Omega)} \quad (f \in L_\sigma^p(\Omega)).$$

*Here, the constants  $C$  are independent of  $t$  and  $f$ .*

As we mentioned before, for the  $L^p$ -theory of the Navier–Stokes equations we pursue two approaches, delivering two types of theorems. The first is derived by following the classical approach of Giga [15]. Note that we take for simplicity  $f = 0$  in (NSE) in this theorem.

**Theorem 1.3.** *Let  $\Omega \subset \mathbb{R}^3$  be a bounded Lipschitz domain. Then there exists  $\varepsilon > 0$  depending only on  $d$  and the Lipschitz character of  $\Omega$  such that for all  $3 \leq r < 3 + \varepsilon$  and all  $a \in L_\sigma^r(\Omega)$  the following statements are valid.*

- (1) *There exists  $T_0 > 0$  and a mild solution  $u : [0, T_0) \rightarrow L_\sigma^r(\Omega)$  to (NSE) with  $f = 0$  and initial velocity  $a$  that satisfies for all  $r \leq p < 3 + \varepsilon$*

$$t \mapsto t^{\frac{3}{2}(\frac{1}{r} - \frac{1}{p})} u(t) \in \text{BC}([0, T_0); L_\sigma^p(\Omega)),$$

$$t \mapsto t^{1 - \frac{3}{2p}} \nabla u(t) \in L^\infty(0, T_0; L^3(\Omega; \mathbb{C}^9)).$$

Moreover,

$$\|u(t) - a\|_{L^r_\sigma(\Omega)} \rightarrow 0 \quad \text{as } t \searrow 0$$

and if  $r < p < 3 + \varepsilon$ , then

$$t^{\frac{3}{2}(\frac{1}{r} - \frac{1}{p})} \|u(t)\|_{L^p_\sigma(\Omega)} \rightarrow 0 \quad \text{as } t \searrow 0.$$

- (2) If  $r > 3$ , there exists a constant  $C > 0$  depending only on  $r$ ,  $p$ , and the constants in Theorem 1.2, such that

$$T_0 \geq C \|a\|_{L^r_\sigma(\Omega)}^{-\frac{4r}{r-3}}.$$

- (3) For all  $3 \leq p < 3 + \varepsilon$  there are positive constants  $C_1, C_2 > 0$  depending only on  $p$  and the constants in Theorem 1.2 such that if  $\|a\|_{L^3_\sigma(\Omega)} < C_1$  the mild solution is global, i.e.,  $T_0 = \infty$ . Moreover, this solution satisfies the estimates

$$\begin{aligned} \|u(t)\|_{L^p_\sigma(\Omega)} &\leq C_2 t^{-\frac{3}{2}(\frac{1}{3} - \frac{1}{p})} \quad (0 < t < \infty) \\ \|\nabla u(t)\|_{L^p(\Omega; \mathbb{C}^9)} &\leq C_2 t^{\frac{3}{2p}-1} \quad (0 < t < \infty). \end{aligned}$$

The final result uses the approach via maximal  $L^q$ -regularity of the Stokes operator in order to obtain strong solutions to the Navier-Stokes equations.

**Theorem 1.4.** *Let  $\Omega \subset \mathbb{R}^3$  be a bounded Lipschitz domain. Then there exists  $\varepsilon > 0$  depending only on  $d$  and the Lipschitz character of  $\Omega$  such that for all  $2 \leq p < 3 + \varepsilon$  and all numbers  $q$  satisfying  $2 \leq q < \infty$  if  $p = 2$  and*

$$\frac{2(p+1)}{3} < q < \infty$$

*if  $2 < p < 3 + \varepsilon$  the following statement is valid: There exists  $C > 0$  such that for all initial velocities  $a$  in the real interpolation space  $(L^p_\sigma(\Omega), \mathcal{D}(A_p))_{1-1/q, q}$  and all  $f \in L^q(0, \infty; L^p(\Omega; \mathbb{C}^3))$  with*

$$\|a\|_{(L^p_\sigma(\Omega), \mathcal{D}(A_p))_{1-1/q, q}} + \|f\|_{L^q(0, \infty; L^p(\Omega; \mathbb{C}^3))} < C$$

*there exists a strong solution  $u$  to (NSE) in the space*

$$W^{1,q}(0, \infty; L^p_\sigma(\Omega)) \cap L^q(0, \infty; \mathcal{D}(A_p)).$$

If  $\Omega$  is bounded and smooth, then Theorem 1.1 is known due to Giga [14] and Theorems 1.2 and 1.3 are known due to Giga [15]. Finally, for a theorem in the fashion of Theorem 1.4 see Amann [1] or Giga and Sohr [16].

This article is organized as follows. In Section 2 we provide the required notation as well as some important facts about the Stokes operator on bounded Lipschitz domains. Section 3 is concerned with the proofs of Theorems 1.1 and 1.2. The final Section 4 is split into two parts. Here, the first part deals with Theorem 1.3 and the second with Theorem 1.4. A precise definition of the notion “solution” is given in the respective subsections.

**Acknowledgments.** I would like to thank Robert Haller-Dintelmann for the supervision and the support during the time of my PhD-studies.

## 2. NOTATION AND PRELIMINARY RESULTS

First, we fix some notation. In the whole article, the space dimension of the underlying Euclidean space will be  $d \geq 3$ . An open set  $\Omega \subset \mathbb{R}^d$  will be called a bounded Lipschitz domain if the boundary can locally be expressed as the graph of a Lipschitz continuous function. The domain of a linear operator  $A$  on a Banach space  $X$  is denoted by  $\mathcal{D}(A)$  and for an interval  $I \subset \mathbb{R}$  we write  $BC(I; X)$  for all bounded and continuous functions with values in  $X$ . The notation for Lebesgue and Sobolev space is as usual. Bessel potential spaces will be denoted by  $H^{s,p}$ . For two

vectors  $x, y \in \mathbb{R}^d$  the tensor product of  $x$  and  $y$  is denoted by  $x \otimes y$  and is given by the matrix which arises by calculating  $xy^T$ , where the superscript  $T$  denotes the transpose of a vector.

For an open and bounded set  $\Omega \subset \mathbb{R}^d$  and  $1 < p < \infty$  the  $L^p$ -space of solenoidal vector fields is defined by

$$L_\sigma^p(\Omega) := \overline{C_{c,\sigma}^\infty(\Omega)}^{L^p},$$

where

$$C_{c,\sigma}^\infty(\Omega) := \{\varphi \in C_c^\infty(\Omega; \mathbb{C}^d) : \operatorname{div}(\varphi) = 0\}.$$

The Sobolev spaces of solenoidal vector fields are defined by

$$W_{0,\sigma}^{1,p}(\Omega) := \overline{C_{c,\sigma}^\infty(\Omega)}^{W^{1,p}}$$

and the space of  $L^p$ -integrable gradient fields is defined by

$$\nabla W^{1,p}(\Omega) := \{\nabla u : u \in W^{1,p}(\Omega)\}.$$

Because  $L_\sigma^2(\Omega)$  is a closed subspace of  $L^2(\Omega; \mathbb{C}^d)$  it is clear that the orthogonal projection  $\mathbb{P}_2$  from  $L^2(\Omega; \mathbb{C}^d)$  onto  $L_\sigma^2(\Omega)$  exists and is bounded. This projection is called Helmholtz projection. The boundedness of the Helmholtz projection on  $L^p$ -spaces for  $p$  in an open interval about two is a well-known result of Fabes, Mendez, and Mitrea [9, Thm. 11.1] and is stated in the following theorem.

**Theorem 2.1** (Fabes, Mendez & Mitrea). *Let  $\Omega \subset \mathbb{R}^d$  be a bounded Lipschitz domain. There exists  $\varepsilon > 0$  such that for all*

$$\frac{3}{2} - \varepsilon < p < 3 + \varepsilon$$

*the Helmholtz projection restricts/extends to a bounded projection  $\mathbb{P}_p$  on  $L^p(\Omega; \mathbb{C}^d)$  with range  $L_\sigma^p(\Omega)$ . Moreover, the range of  $\operatorname{Id} - \mathbb{P}_p$  is given by  $\nabla W^{1,p}(\Omega)$ .*

The Stokes operator  $A_2$  on an open and bounded set  $\Omega \subset \mathbb{R}^d$  is defined by means of Kato's form method as the  $L_\sigma^2(\Omega)$ -realization of the sesquilinear form

$$\mathfrak{a} : W_{0,\sigma}^{1,2}(\Omega) \times W_{0,\sigma}^{1,2}(\Omega) \rightarrow \mathbb{C}, \quad (u, v) \mapsto \int_\Omega \nabla u \cdot \overline{\nabla v} \, dx.$$

By symmetry and coercivity of the form it is clear that  $A_2$  is self-adjoint and that  $-A_2$  generates a bounded analytic semigroup. This semigroup is denoted by  $e^{-tA_2}$  and is called the Stokes semigroup. For a clear discussion of the facts above, see [23, Sec. 4]. In particular, Mitrea and Monniaux give in [23, Thm. 4.7] the following convenient characterization of the Stokes operator.

**Theorem 2.2.** *If  $\Omega \subset \mathbb{R}^d$  is a bounded Lipschitz domain, then the Stokes operator on  $L_\sigma^2(\Omega)$  is characterized by*

$$\begin{aligned} \mathcal{D}(A_2) &= \{u \in W_{0,\sigma}^{1,2}(\Omega) : \exists \pi \in L^2(\Omega) \text{ such that } -\Delta u + \nabla \pi \in L_\sigma^2(\Omega)\} \\ A_2 u &= -\Delta u + \nabla \pi. \end{aligned}$$

Note that in the theorem above, “ $-\Delta u + \nabla \pi$ ” has to be understood in the sense of distributions.

To define the Stokes operator on the spaces  $L_\sigma^p(\Omega)$ , one distinguishes the cases  $p > 2$  and  $p < 2$ . If  $p > 2$ , then the Stokes operator  $A_p$  on  $L_\sigma^p(\Omega)$  is defined as the part of  $A_2$  in  $L_\sigma^p(\Omega)$ , i.e.,

$$\mathcal{D}(A_p) := \{u \in \mathcal{D}(A_2) : A_2 u \in L_\sigma^p(\Omega)\}, \quad A_p u := A_2 u.$$

If  $p < 2$  and if  $A_2$  is closable in  $L_\sigma^p(\Omega)$ , then the Stokes operator  $A_p$  is defined as the closure of  $A_2$  in  $L_\sigma^p(\Omega)$ , i.e.,

$$\begin{aligned} \mathcal{D}(A_p) &:= \{u \in L_\sigma^p(\Omega) : \exists f \in L_\sigma^p(\Omega), \exists (u_n)_{n \in \mathbb{N}} \subset \mathcal{D}(A_2) \text{ such that } u_n \rightarrow u \text{ and } A_2 u_n \rightarrow f \text{ in } L^p\} \\ A_p u &:= f. \end{aligned}$$

**Remark 2.3.** If  $2 < p < \infty$  is such that the Helmholtz projection is bounded on  $L^p(\Omega; \mathbb{C}^d)$  then one can show as in [34, Prop. 5.2.16] that  $A_p$  is densely defined if and only if  $A_2$  is closable in  $L^p_\sigma(\Omega)$ , where  $p'$  is the Hölder conjugate exponent of  $p$ . Moreover, if this applies then  $A_p^* = A_{p'}$ , where  $A_p^*$  denotes the dual operator to  $A_p$ .

In 2012, Shen proved in his seminal paper [28] the following result.

**Theorem 2.4** (Shen). *Let  $\Omega \subset \mathbb{R}^d$  be a bounded Lipschitz domain. Then there exists  $\varepsilon > 0$  depending only on  $d$  and the Lipschitz character of  $\Omega$  such that for all*

$$\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{1}{d} + \varepsilon$$

*$A_p$  is sectorial of angle 0. In particular,  $-A_p$  generates a bounded analytic semigroup with  $0 \in \rho(A_p)$  and  $A_p$  is closed and densely defined.*

Let us quantify Shen's statement that the Stokes operator is densely defined. For this purpose, define for  $1 < p < \infty$

$$W_{0,\sigma}^{2,p}(\Omega) := \overline{C_{c,\sigma}^\infty(\Omega)}^{W^{2,p}}.$$

Then the following lemma is valid.

**Lemma 2.5.** *Let  $\Omega \subset \mathbb{R}^d$  be a bounded Lipschitz domain. Then there exists  $\varepsilon > 0$  such that for all*

$$\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{1}{d} + \varepsilon,$$

*the space  $W_{0,\sigma}^{2,p}(\Omega)$  is embedded continuously into  $\mathcal{D}(A_p)$ . In particular, the representation formula*

$$(2.1) \quad A_p u = -\mathbb{P}_p \Delta u \quad (u \in W_{0,\sigma}^{2,p}(\Omega))$$

*is valid.*

*Proof.* We distinguish the cases  $p \geq 2$  and  $p < 2$ . If  $p \geq 2$ , let  $u \in W_{0,\sigma}^{2,p}(\Omega)$ . Then, by virtue of Theorem 2.1 there exists  $g \in W^{1,p}(\Omega)$  such that  $(\text{Id} - \mathbb{P}_p)\Delta u = \nabla g$ . Knowing this, calculate

$$-\Delta u = -\mathbb{P}_p \Delta u - (\text{Id} - \mathbb{P}_p)\Delta u = -\mathbb{P}_p \Delta u - \nabla g.$$

Since  $\Omega$  is bounded, Theorem 2.2 gives  $u \in \mathcal{D}(A_2)$ . The definition of  $\mathcal{D}(A_p)$  then delivers  $u \in \mathcal{D}(A_p)$ . In particular, (2.1) is valid.

If  $p < 2$ , let  $u \in W_{0,\sigma}^{2,p}(\Omega)$  and  $(u_n)_{n \in \mathbb{N}} \subset C_{c,\sigma}^\infty(\Omega)$  be an appropriate sequence that approximates  $u$  in  $W^{2,p}(\Omega; \mathbb{C}^d)$ . Since  $\mathbb{P}_2$  extends to a bounded operator on  $L^p(\Omega; \mathbb{C}^d)$  by Theorem 2.1, the representation formula (2.1) for  $A_2$  shows that  $(A_2 u_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $L^p_\sigma(\Omega)$ . Consequently,  $u \in \mathcal{D}(A_p)$  since  $A_p$  is the closure of  $A_2$  in  $L^p_\sigma(\Omega)$ . Additionally, we find that

$$A_p u = -\mathbb{P}_p \Delta u.$$

Finally, the continuous embedding follows by the boundedness of  $\mathbb{P}_p$  and the representation formula (2.1).  $\square$

We close this section by mentioning some functional analytic facts of the Stokes operator on  $L^p_\sigma(\Omega)$ . We start by introducing the notion of maximal regularity.

Let  $X$  be a Banach space and  $-A$  be the generator of a bounded analytic semigroup on  $X$ . For  $1 < q < \infty$ ,  $f \in L^q(0, \infty; X)$ , and  $a$  in the real interpolation space  $(X, \mathcal{D}(A))_{1-1/q, q}$  consider the problem

$$(ACP) \quad \begin{cases} u'(t) + Au(t) = f(t) & 0 < t < \infty \\ u(0) = a. \end{cases}$$

It is classical that (ACP) admits a unique mild solution, which can be written via the variations of constants formula as

$$u(t) = e^{-tA}a + \int_0^t e^{-(t-s)A} f(s) \, ds,$$

see, e.g., [2, Prop. 3.1.16]. Now, we say that  $A$  has *maximal  $L^q$ -regularity* if for each  $f \in L^q(0, \infty; X)$  and  $a \in (X, \mathcal{D}(A))_{1-1/q, q}$  the solution  $u$  is almost everywhere differentiable, satisfies  $u(t) \in \mathcal{D}(A)$  for almost every  $0 < t < \infty$ , and if

$$u', Au \in L^q(0, \infty; X).$$

It is well-known, see [5, 19], that one gets an estimate of the  $L^q$ -norms of  $u'$  and  $Au$  by employing the closed graph theorem. More precisely, there exists a constant  $C > 0$  such that

$$(2.2) \quad \|u'\|_{L^q(0, \infty; X)} + \|Au\|_{L^q(0, \infty; X)} \leq C \{ \|f\|_{L^q(0, \infty; X)} + \|a\|_{(X, \mathcal{D}(A))_{1-1/q, q}} \}.$$

Finally, it is worth noting that maximal  $L^q$ -regularity is independent of  $q$ . This means that  $A$  has maximal  $L^q$ -regularity for some  $q \in (1, \infty)$  if and only if it has maximal  $L^q$ -regularity for all  $q \in (1, \infty)$ , see [5]. This property will be important in Subsection 4.2 in order to find the “right” spaces to perform a fixed point argument.

For better readability, define the spaces

$$\mathbb{E} := W^{1,q}(0, \infty; X) \cap L^q(0, \infty; \mathcal{D}(A)), \quad \mathbb{F} := L^q(0, \infty; X) \times (X, \mathcal{D}(A))_{1-1/q, q}$$

endowed with the canonical norms. Then, if additionally  $0 \in \rho(A)$ , (2.2) turns into

$$(2.3) \quad \|u\|_{\mathbb{E}} \leq C \|(f, a)\|_{\mathbb{F}}.$$

The following theorem concerns the maximal  $L^q$ -regularity of the Stokes operator and can be found in [20, Prop. 13] or [34, Thm. 5.2.24].

**Theorem 2.6** (Kunstmann & Weis). *Let  $\Omega \subset \mathbb{R}^d$  be a bounded Lipschitz domain and  $1 < q < \infty$ . Then there exists  $\varepsilon > 0$  depending only on  $d$  and the Lipschitz character of  $\Omega$  such that for all*

$$\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{1}{d} + \varepsilon$$

*the Stokes operator  $A_p$  has maximal  $L^q$ -regularity.*

Since  $A_p$  is injective and sectorial of angle 0, one can follow the construction in Haase [17, Ch. 2] to assign a linear and closed operator  $f(A_p)$  to each holomorphic function  $f : S_\theta \rightarrow \mathbb{C}$  exhibiting at most polynomial growth at 0 and infinity, where for  $\theta \in (0, \pi)$

$$S_\theta := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \theta\}.$$

In particular, one can assign an operator to functions  $f \in H^\infty(S_\theta)$ , which is the algebra of bounded and holomorphic functions on  $S_\theta$ , and to functions of the form

$$S_\theta \ni z \mapsto z^\alpha$$

for each  $\alpha \in \mathbb{C}$ . The latter type of functions lead to fractional powers  $A_p^\alpha$  of the Stokes operator. If for each  $f \in H^\infty(S_\theta)$  the operator  $f(A_p)$  is bounded and if one has the estimate

$$\|f(A_p)\|_{\mathcal{L}(L^p_\sigma(\Omega))} \leq C \|f\|_{L^\infty(S_\theta)} \quad (f \in H^\infty(S_\theta)),$$

then one says that the  $H^\infty$ -calculus of  $A_p$  is bounded. This is exactly what Kunstmann and Weis proved in [20, Thm. 16].

**Theorem 2.7** (Kunstmann & Weis). *Let  $\Omega \subset \mathbb{R}^d$  be a bounded Lipschitz domain. Then there exists  $\varepsilon > 0$  depending only on  $d$  and the Lipschitz character of  $\Omega$  such that for all*

$$\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{1}{d} + \varepsilon$$

*the  $H^\infty$ -calculus of the Stokes operator  $A_p$  is bounded.*

The boundedness of the  $H^\infty$ -calculus of  $A_p$  gives information about the domains of the fractional powers of  $A_p$ . Indeed, if  $p$  is in the same range as in the preceding theorem, these can be computed by complex interpolation

$$(2.4) \quad [L_\sigma^p(\Omega), \mathcal{D}(A_p)]_\alpha = \mathcal{D}(A_p^\alpha) \quad (\alpha \in (0, 1)),$$

see [17, Thm. 6.6.9]. This property will be crucial in the following section.

### 3. THE SQUARE ROOT OF $A_p$ AND MAPPING PROPERTIES OF THE SEMIGROUP

With the considerations of Section 2 we can prove Theorem 1.1. Note that this proof is motivated by a calculation Shen performed in [27, Lem. 3.5].

**Proof of Theorem 1.1.** To obtain the embedding  $W_{0,\sigma}^{1,p}(\Omega) \subset \mathcal{D}(A_p^{1/2})$  combine (2.4) together with Lemma 2.5 to get the continuous embedding

$$[L_\sigma^2(\Omega), W_{0,\sigma}^{2,p}(\Omega)]_{1/2} \subset [L_\sigma^p(\Omega), \mathcal{D}(A_p)]_{1/2} = \mathcal{D}(A_p^{1/2}).$$

Now, the desired embedding follows since the interpolation space on the left-hand side is known [23, Prop. 2.10, Thm. 2.12] to be

$$[L_\sigma^2(\Omega), W_{0,\sigma}^{2,p}(\Omega)]_{1/2} = W_{0,\sigma}^{1,p}(\Omega).$$

To obtain the opposite embedding, let  $F \in C_c^\infty(\Omega; \mathbb{C}^{d \times d})$  and let  $p'$  be the Hölder conjugate exponent to  $p$ . Consider the Stokes problem

$$\begin{cases} -\Delta u + \nabla \pi = \mathbb{P}_{p'} \operatorname{div}(F) & \text{in } \Omega \\ \operatorname{div}(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Since  $(\mathbb{P}_{p'} - \operatorname{Id}) \operatorname{div}(F) = \nabla g$  for some  $g \in W^{1,p'}(\Omega)$  by Theorem 2.1,  $u$  and  $\pi - g$  solve the Stokes problem with right-hand side being  $\operatorname{div}(F)$ . Since  $\operatorname{div}(F)$  induces a functional in  $(W_0^{1,p}(\Omega; \mathbb{C}^d))^*$  obeying the estimate

$$\|\operatorname{div}(F)\|_{(W_0^{1,p})^*} \leq \|F\|_{L^{p'}(\Omega; \mathbb{C}^{d \times d})},$$

we find by [24, Thm. 10.15], see also [13, Thm. 1.3] in the case where  $\partial\Omega$  is connected, that

$$\|\nabla A_{p'}^{-1} \mathbb{P}_{p'} \operatorname{div}(F)\|_{L^{p'}(\Omega; \mathbb{C}^{d^2})} = \|\nabla u\|_{L^{p'}(\Omega; \mathbb{C}^{d^2})} \leq C \|F\|_{L^{p'}(\Omega; \mathbb{C}^{d \times d})}.$$

Here, the constant  $C > 0$  is independent of  $F$ . Appealing to the first part of this proof and to Poincaré's inequality

$$\begin{aligned} \|A_{p'}^{-1/2} \mathbb{P}_{p'} \operatorname{div}(F)\|_{L_\sigma^{p'}(\Omega)} &= \|A_{p'}^{1/2} A_{p'}^{-1} \mathbb{P}_{p'} \operatorname{div}(F)\|_{L_\sigma^{p'}(\Omega)} \\ &\leq C \|\nabla A_{p'}^{-1} \mathbb{P}_{p'} \operatorname{div}(F)\|_{L^{p'}(\Omega; \mathbb{C}^{d^2})} \\ &\leq C \|F\|_{L^{p'}(\Omega; \mathbb{C}^{d \times d})}. \end{aligned}$$

By duality, one then derives the continuous embedding  $\mathcal{D}(A_p^{1/2}) \subset W_{0,\sigma}^{1,p}(\Omega)$ .  $\square$

Having determined the domain of the square root of the Stokes operator, the proof of Theorem 1.2 is fairly short.

**Proof of Theorem 1.2.** We choose  $\varepsilon$  as in the statement of Theorem 1.1. Then for

$$\left| \frac{1}{p} - \frac{1}{2} \right| < \frac{1}{d} + \varepsilon$$

one finds by Theorem 1.1 and since  $0 \in \rho(A_p)$

$$\|\nabla e^{-tA_p} f\|_{L^p(\Omega; \mathbb{C}^{d^2})} \leq C \|A_p^{1/2} e^{-tA_p} f\|_{L_\sigma^p(\Omega)} \leq C t^{-1/2} \|f\|_{L_\sigma^p(\Omega)} \quad (f \in L_\sigma^p(\Omega), t > 0).$$

Note that the estimate  $\|A^{1/2}e^{-tA}x\|_X \leq Ct^{-1/2}\|x\|_X$  is valid for all generators  $-A$  of bounded analytic semigroups on a Banach space  $X$ , see [17, Prop. 3.4.3].

The rest of the proof is concerned with deducing the  $L^p$ - $L^q$ -mapping properties of the Stokes semigroup from the gradient estimates. Since  $\mathcal{D}(A_p) \subset W_0^{1,p}(\Omega; \mathbb{C}^d)$ , we can employ Gagliardo–Nirenberg’s inequality, [25, p. 125], together with the first part of this proof and the boundedness of the semigroup (Theorem 2.4) to deduce

$$\|e^{-tA_p}f\|_{L_\sigma^q(\Omega)} \leq C\|\nabla e^{-tA_p}f\|_{L^p(\Omega; \mathbb{C}^{d^2})}^\alpha \|e^{-tA_p}f\|_{L_\sigma^p(\Omega)}^{1-\alpha} \leq Ct^{-\alpha/2}\|f\|_{L_\sigma^p(\Omega)},$$

where  $\alpha$  is given by the conditions

$$0 \leq \alpha \leq 1 \quad \text{and} \quad \frac{1}{q} = \left(\frac{1}{p} - \frac{1}{d}\right)\alpha + \frac{1-\alpha}{p}.$$

A calculation of  $\alpha$  reveals the desired exponent for  $t$ . However, the condition  $\alpha \leq 1$  imposes the restriction

$$\frac{1}{p} - \frac{1}{q} \leq \frac{1}{d}.$$

This restriction can be avoided by writing  $e^{-tA_p} = (e^{-t/nA_p})^n$  for some suitable  $n \in \mathbb{N}$  and then by employing the  $L^p$ - $L^q$ -estimate above iteratively.  $\square$

#### 4. EXISTENCE THEORY TO THE NAVIER–STOKES EQUATIONS

When dealing with the Navier–Stokes equations, the space dimension  $d$  will be assumed to be 3 in this section.

**4.1. Solvability in the critical space  $L^\infty(0, T; L_\sigma^3(\Omega))$  via an iteration scheme.** We start this section with a brief comparison of the results of Giga [15, Thm. 4] and Kato [18, Thm. 1, Thm. 2]. These theorems state the local existence of solutions in  $L^\infty(0, T; L_\sigma^3(\Omega))$  for initial data of arbitrary size in  $L_\sigma^3(\Omega)$  as well as global existence for small initial data. For simplicity, we will set the right-hand side  $f$  in (NSE) to zero and we denote the Stokes operator  $A_p$  by  $A$  and  $\mathbb{P}_p$  by  $\mathbb{P}$ . Denoting the application of  $\mathbb{P}$  to the nonlinear term of (NSE) by  $F[u, u]$ , both authors constructed solutions to the Navier–Stokes equations by means of a successive approximation defined by

$$(4.1) \quad u_0(t) := e^{-tA}a$$

$$(4.2) \quad u_{j+1}(t) := u_0(t) + Su_j(t) := u_0(t) - \int_0^t e^{-(t-s)A}F[u_j(s), u_j(s)] \, ds \quad (j \in \mathbb{N}).$$

It was proven that  $(u_j)_{j \in \mathbb{N}}$  converges in a suitable sense, so that one can take the limit in (4.2) to obtain a function  $u$ , which satisfies the variation of constants formula

$$(4.3) \quad u(t) = u_0(t) - \int_0^t e^{-(t-s)A}F[u(s), u(s)] \, ds.$$

For a fixed time  $T > 0$  and an initial datum  $a \in L_\sigma^r(\Omega)$  with  $r \geq 3$ , we say that a continuous function  $u : [0, T] \rightarrow L_\sigma^r(\Omega)$  is called a mild solution to (NSE) if it satisfies (4.3).

The first difference in the approaches of Giga and Kato lies in the interpretation of the nonlinearity. While Kato left it unchanged, i.e.,

$$F[u, u] = \mathbb{P}(u \cdot \nabla)u,$$

Giga used the well-known fact that  $F[u, u]$  can be expressed in divergence-form if  $u$  is divergence-free. More precisely, Giga wrote the nonlinearity in the form

$$(4.4) \quad F[u, u] = \mathbb{P} \operatorname{div}(u \otimes u).$$



To prove the convergence of  $(u_j)_{j \in \mathbb{N}}$ , Kato used merely the validity of the following two types of estimates

- (1)  $\|e^{-tA}f\|_{L^q_\sigma(\Omega)} \leq Ct^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}\|f\|_{L^p_\sigma(\Omega)}$  for all  $t > 0$  and  $f \in L^p_\sigma(\Omega)$ ;
- (2)  $\|\nabla e^{-tA}f\|_{L^p(\Omega; \mathbb{C}^9)} \leq Ct^{-\frac{1}{2}}\|f\|_{L^p_\sigma(\Omega)}$  for all  $t > 0$  and  $f \in L^p_\sigma(\Omega)$ ,

which hold for all  $1 < p \leq q < \infty$  if  $\Omega = \mathbb{R}^3$  or if  $\Omega$  is a bounded and smooth domain. In comparison to that, Giga used the validity of (1) and replaced (2) by the fact that  $e^{-tA}\mathbb{P} \operatorname{div}$  a priori defined on  $W_0^{1,p}(\Omega; \mathbb{C}^{3 \times 3})$  extends to a bounded operator from  $L^p(\Omega; \mathbb{C}^{3 \times 3})$  into  $L^p_\sigma(\Omega)$  with the estimate

- (2')  $\|e^{-tA}\mathbb{P} \operatorname{div} F\|_{L^p_\sigma(\Omega)} \leq Ct^{-\frac{1}{2}}\|F\|_{L^p(\Omega; \mathbb{C}^{3 \times 3})}$  for all  $t > 0$  and  $F \in L^p(\Omega; \mathbb{C}^{3 \times 3})$ .

Since (2) and (2') are dual to each other, both authors used the same estimates for their proof. However, Kato writes in [18, Eq. (2.4)] that his proof works not only for the Navier-Stokes equations but for all equations where the corresponding semigroup satisfies (1) and (2), and where the nonlinear term satisfies the Hölder type inequality

$$\|F[u, v]\|_{L^s(\Omega; \mathbb{C}^3)} \leq C\|u\|_{L^p(\Omega; \mathbb{C}^3)}\|\nabla v\|_{L^q(\Omega; \mathbb{C}^9)} \quad \left(\frac{1}{s} = \frac{1}{p} + \frac{1}{q}\right).$$

Comparing [18, Thm. 1, Thm. 2] and [15, Thm. 4] one readily verifies that Kato proves not only a control on  $u$  but also on  $\nabla u$ , whereas Giga obtains only control on  $u$ . This seems to be odd because both authors use the very same estimates and Giga even uses more structural information of the nonlinearity, as he needs to rewrite it into the form (4.4). In conclusion, one could guess that Giga's proof should either yield a stronger result than Kato's or that one can weaken the required estimates while obtaining the same result. In this subsection, we will show that the latter is indeed true.

Note that on a general bounded Lipschitz domain, the Stokes semigroup does not exist on  $L^p_\sigma(\Omega)$  for all  $1 < p < \infty$ , see Deuring [7]. Furthermore, Theorem 1.2 provides us with the important estimates only in an interval about 2. So, we will have to review Giga's proof of [15, Thm. 1] under these restrictions and also under the viewpoint to weaken the requirement of (2'). To do so, we introduce the more abstract framework in which Giga proved [15, Thm. 1].

Consider a semilinear parabolic problem of the type

$$(SPP) \quad \begin{cases} \partial_t u(t) + Au(t) = F[u](t) & (0 < t < T) \\ u(0) = a, \end{cases}$$

where  $F[u]$  represents the nonlinear part of the equation and  $-A$  is the generator of an analytic semigroup on a range of certain subspaces of  $L^p$ .

Let  $\Omega \subset \mathbb{R}^d$ ,  $d \geq 1$ , be a locally compact Hausdorff space endowed with a  $\sigma$ -finite, complete Radon measure  $\mu$  whose  $\sigma$ -algebra contains the Borel sets. Let further  $l, k \in \mathbb{N}$  and  $1 < p_- < p^\circ < p_+ < \infty$  be real numbers. For  $p_- < p < p_+$ , let  $U^p$  be a closed subspace of  $L^p(\Omega, \mu; \mathbb{C}^l)$ , and let  $P_p$  be a continuous projector from  $L^p(\Omega, \mu; \mathbb{C}^l)$  onto  $U^p$  such that the restriction of  $P_p$  to  $C_c(\Omega; \mathbb{C}^l)$  is independent of  $p$ .

We assume the following properties of the quantities arising in (SPP). For brevity, we will write  $\|f\|_{U^p}$  instead of  $\|f\|_{L^p(\Omega, \mu; \mathbb{C}^l)}$  if  $f \in U^p$ .

**Assumption 4.1.** (A) For all  $p_- < p < p_+$  there are operators  $-A_p : \mathcal{D}(A_p) \subset U^p \rightarrow U^p$  that generate analytic semigroups of operators  $(e^{-tA_p})_{t \geq 0}$  on  $U^p$ , which are consistent on the  $U^p$ -scale, i.e.,  $e^{-tA_{p_1}}$  coincides with  $e^{-tA_{p_2}}$  on  $U^{p_1} \cap U^{p_2}$  for all  $t \geq 0$  and  $p_- < p_1, p_2 < p_+$ . Moreover, there are constants  $n, m \geq 1$  such that for every fixed  $0 < T < \infty$  there exists  $M_T > 0$  such that for all  $f \in U^p$  and  $0 < t < T$  the estimate

$$\|e^{-tA_p}f\|_{U^q} \leq M_T t^{-\sigma} \|f\|_{U^p}$$

holds with  $\sigma := \frac{n}{m}(\frac{1}{p} - \frac{1}{q})$  and  $p_- < p \leq q < p_+$ .

- (N) There exist constants  $\alpha > 0$  and  $0 \leq \gamma < m$  such that for each  $p_- < p \leq p^\circ$  the nonlinear term  $F[u]$  can be decomposed into  $F[u] = \Gamma Gu$  with  $\Gamma$  and  $G$  having the following properties. The operator  $\Gamma : \mathcal{D}(\Gamma) \subset L^p(\Omega, \mu; \mathbb{C}^k) \rightarrow U^p$  is linear and densely defined. Furthermore, for each  $t > 0$  and  $p^\circ \leq q < p_+$  the operator  $e^{-tA_p} \Gamma$  extends to a bounded operator from  $L^p(\Omega, \mu; \mathbb{C}^k)$  into  $L^q(\Omega, \mu; \mathbb{C}^l)$ , such that for each given  $0 < T < \infty$  there exists a constant  $N_{1,T} > 0$  such that for all  $f \in L^p(\Omega, \mu; \mathbb{C}^k)$  and all  $0 < t < T$

$$\|e^{-tA_p} \Gamma f\|_{U^q} \leq N_{1,T} t^{-\sigma - \frac{\gamma}{m}} \|f\|_{L^p(\Omega, \mu; \mathbb{C}^k)},$$

where  $m$  and  $\sigma$  are the same numbers as in (A).

The operator  $G$  is supposed to reflect the nonlinear part and satisfies  $G0 = 0$ . Moreover, there exists a constant  $N_2$  such that

$$\|Gv - Gw\|_{L^p(\Omega, \mu; \mathbb{C}^k)} \leq N_2 \|v - w\|_{U^{p(1+\alpha)}} (\|v\|_{U^{p(1+\alpha)}}^\alpha + \|w\|_{U^{p(1+\alpha)}}^\alpha)$$

for all  $v, w \in U^{p(1+\alpha)}$ .

Heuristically,  $\Gamma$  has the role of a differential operator of order  $\gamma$  and  $Gu$  behaves like  $|u|^\alpha u$ .

**Remark 4.2.** In comparison to Giga [15], we carried out three changes in these assumptions. First of all, we introduced the numbers  $p_-$ ,  $p^\circ$ , and  $p_+$  and assume that (A) only holds between  $p_-$  and  $p_+$ . Giga assumed this validity for all  $1 < p < \infty$ .

Secondly, we replaced the first estimate in (N). In [15], this estimate reads as

$$(4.5) \quad \|e^{-tA_p} \Gamma f\|_{U^p} \leq N_{1,T} t^{-\frac{\gamma}{m}} \|f\|_{L^p(\Omega, \mu; \mathbb{C}^k)}$$

and is assumed for all  $1 < p < \infty$ . This is the same estimate as in (N) by taking  $p = q$ . However, recalling the assumptions in (N), this is only possible if  $p = p^\circ = q$ , i.e., only for one special choice of Lebesgue exponents. Note that (4.5) together with (A) imply the validity of

$$\|e^{-tA_p} \Gamma f\|_{U^q} \leq N_{1,T} t^{-\sigma - \frac{\gamma}{m}} \|f\|_{L^p(\Omega, \mu; \mathbb{C}^k)}$$

by writing  $e^{-tA_p} \Gamma f = e^{-\frac{t}{2}A_p} e^{-\frac{t}{2}A_p} \Gamma f$ .

It is the main insight of our analysis of Giga's proof, that (4.5) is never employed alone, but always in combination with the estimates of (A). This leads to the weaker estimate stated in Assumption 4.1 (N).

Thirdly, Giga assumed that the operator  $\Gamma$  should be closed from  $L^p(\Omega, \mu; \mathbb{C}^k)$  to  $U^q$  for some  $q$ . However, it turns out that this assumption is not needed. Mainly because  $\Gamma$  will never be considered as an operator alone, but always in combination with the semigroup that is assumed to extend to a bounded operator defined on all of  $L^p(\Omega, \mu; \mathbb{C}^k)$ .

**Assumption 4.3.** With  $p_-$ ,  $p_+$ , and  $\alpha$  as above, assume that

$$\max\{p_-(1+\alpha), p^\circ\} < \min\{p_+, p^\circ(1+\alpha)\}.$$

Recall the iteration scheme defined in (4.1) and (4.2) for the Navier–Stokes equations. We will use the same notation in this more general context.

**Theorem 4.4.** *Under Assumptions 4.1 and 4.3 the following holds true for every  $T > 0$ .*

- (1) Let  $p_0, p'_0$  denote

$$p_0 := \frac{n\alpha}{m - \gamma} \quad \text{and} \quad p'_0 := \max \left\{ p_0, p^\circ, \frac{np^\circ(1+\alpha)}{n + p^\circ m} \right\}.$$

Then,  $p_0 = p'_0$  holds if  $p_0 \geq p^\circ$ . Suppose that  $a \in U^r$  for a fixed

$$p'_0 < r < \min\{p_+, p^\circ(1+\alpha)\}$$

or, if  $p_0 > p^\circ$ ,  $r$  could even equal to  $p_0$ . Then there is  $0 < T_0 \leq T$  and a function  $u$  on  $[0, T_0]$  that satisfies

$$\begin{aligned} t \mapsto t^\sigma u(t) &\in \text{BC}([0, T_0]; \text{U}^p) && \text{for } r \leq p < \min\{p_+, p^\circ(1 + \alpha)\}, \\ t^\sigma \|u(t)\|_{\text{U}^p} &\rightarrow 0 \quad \text{as } t \rightarrow 0 && \text{for } r < p < \min\{p_+, p^\circ(1 + \alpha)\}, \end{aligned}$$

whenever  $\sigma := \frac{n}{m}(\frac{1}{r} - \frac{1}{p})$  satisfies  $0 \leq \sigma < \frac{1}{1+\alpha}$ . Moreover, for every

$$\max\{r, p_-(1 + \alpha)\} < p < \min\{p_+, p^\circ(1 + \alpha)\} \quad \text{and} \quad h := \frac{p}{1 + \alpha},$$

$u$  solves the equation

$$(4.6) \quad u(t) = e^{-tA_r} a + \int_0^t e^{-(t-\tau)A_h} \Gamma G u(\tau) \, d\tau \quad (t \in (0, T_0)).$$

(2) If  $r > p'_0$ , the lifespan has the lower bound

$$(4.7) \quad T_0 \geq C \|a\|_{\text{U}^r}^{-\frac{\alpha}{1-\beta(r)}}$$

with  $\beta(r) := \frac{1}{m}(\gamma + \frac{n\alpha}{r})$  and  $C > 0$  depending only on  $\alpha, M_T, N_{1,T}, N_2, \gamma, n, m, r$ , and  $p$ .

(3) If  $p_0 > p^\circ$  and

$$p_0 \leq p < \min\{p_+, p^\circ(1 + \alpha)\} \quad \text{with} \quad \frac{n}{m} \left( \frac{1}{p_0} - \frac{1}{p} \right) < \frac{1}{1 + \alpha},$$

there is a positive constant  $\mathcal{C}$  such that if  $\|a\|_{\text{U}^{p_0}} < \mathcal{C}$ , then  $T_0$  equals  $T$ . Furthermore, if  $M_T$  and  $N_{1,T}$  are uniformly bounded in  $0 < T < \infty$ , we can take  $T = \infty$  and get

$$\|u(t)\|_{\text{U}^p} \leq C t^{-\sigma}, \quad (0 < t < \infty),$$

with  $C$  and  $\mathcal{C}$  depending only on  $\alpha, M_T, N_{1,T}, N_2, \gamma, n, m, r$ , and  $p$  (if  $T = \infty$  these constants depend on the uniform bounds of  $M_T$  and  $N_{1,T}$ ).

*Proof.* That  $p_0 = p'_0$  holds if  $p_0 \geq p^\circ$  follows by proving that  $p^\circ \mapsto np^\circ(1 + \alpha)/(n + p^\circ m)$  increases and by using  $p^\circ \leq p_0$ .

Next, Assumption 4.1 (N), implies for all

$$p_-(1 + \alpha) < p \leq p^\circ(1 + \alpha), \quad p^\circ \leq s < p_+,$$

and all  $v, w \in \text{U}^p$

$$(4.8) \quad \|e^{-tA_h} (F[v] - F[w])\|_{\text{U}^s} \leq N_{1,T} N_2 t^{\delta - \beta(p)} \|v - w\|_{\text{U}^p} (\|v\|_{\text{U}^p}^\alpha + \|w\|_{\text{U}^p}^\alpha) \quad (0 < t < T).$$

Here,  $\delta := \frac{n}{m}(\frac{1}{s} - \frac{1}{p})$  and  $h$  and  $\beta(p)$  are defined as in the theorem.

Next, fix a number  $0 < T_0 \leq T$  and continue by establishing an a priori estimate for

$$K_j := K_j(T_0) := \sup_{0 < t < T_0} t^\sigma \|u_j(t)\|_{\text{U}^p} \quad (j \geq 0),$$

where  $\sigma = \frac{n}{m}(\frac{1}{r} - \frac{1}{p})$  and  $p$  are chosen such that

$$(4.9) \quad \begin{aligned} &\max \left\{ 0, \frac{n}{m} \left( \frac{1}{r} - \frac{1}{\max\{p_-, \frac{p^\circ}{1+\alpha}\}(1 + \alpha)} \right) \right\} < \sigma \quad \text{and} \\ &\sigma < \min \left\{ \frac{1}{1 + \alpha}, \frac{n}{m} \left( \frac{1}{r} - \frac{1}{\min\{p_+, p^\circ(1 + \alpha)\}} \right) \right\}. \end{aligned}$$

If this choice is possible, this implies

$$p < \min\{p_+, p^\circ(1 + \alpha)\}, \quad p > \max\{p_-(1 + \alpha), p^\circ\}, \quad \text{and} \quad p > r,$$

so that especially  $p > p_0$ . To prove that  $\sigma$  and  $p$  can be chosen subject to (4.9) note that due to  $\alpha > 0$ ,  $r < \min\{p_+, p^\circ(1+\alpha)\}$ , and  $\max\{p_-(1+\alpha), p^\circ\} < \min\{p_+, p^\circ(1+\alpha)\}$  (by Assumption 4.3) it suffices to prove that

$$\frac{n}{m} \left( \frac{1}{r} - \frac{1}{\max\{p_-, \frac{p^\circ}{1+\alpha}\}(1+\alpha)} \right) < \frac{1}{1+\alpha}.$$

If  $r > p'_0$  is satisfied, this follows by

$$p'_0 \geq \frac{np^\circ(1+\alpha)}{n+p^\circ m} \Leftrightarrow \frac{1}{1+\alpha} \geq \frac{n}{m} \left( \frac{1}{p'_0} - \frac{1}{p^\circ(1+\alpha)} \right) \geq \frac{n}{m} \left( \frac{1}{p'_0} - \frac{1}{\max\{p_-, \frac{p^\circ}{1+\alpha}\}(1+\alpha)} \right).$$

If  $p_0 > p^\circ$  and  $r = p_0$  the definition of  $p_0$  shows that

$$\frac{1+\alpha}{p_0} = \frac{1}{p_0} + \frac{m}{n} - \frac{\gamma}{n} \leq \frac{1}{p_0} + \frac{m}{n} < \frac{1}{p^\circ} + \frac{m}{n} \leq \frac{1}{\max\{p_-, \frac{p^\circ}{1+\alpha}\}} + \frac{m}{n}$$

what proves the desired inequality. To establish the a priori bound for  $K_j$  recall the iteration scheme defined via (4.1) and (4.2), i.e.,

$$u_{j+1} = u_0 + Su_j.$$

An application of the auxiliary estimate (4.8) with  $v = u_j$ ,  $w = 0$ , and  $s = p$  to the term  $Su_j$  yields

$$(4.10) \quad t^\sigma \|Su_j(t)\|_{U^p} \leq t^\sigma \int_0^t \frac{N_{1,T}N_2}{(t-\tau)^{\beta(p)}} \|u_j(\tau)\|_{U^p}^{1+\alpha} d\tau;$$

here  $\max\{p_-(1+\alpha), p^\circ\} < p < \min\{p_+, p^\circ(1+\alpha)\}$  is used. Since  $\beta(r) = \beta(p) + \sigma\alpha$ , this gives the iterative estimate

$$K_{j+1} \leq K_0 + N_{1,T}N_2BK_j^{1+\alpha}T_0^{1-\beta(r)}$$

with

$$(4.11) \quad B = \int_0^1 \frac{1}{(1-\tau)^{\beta(p)}} \frac{1}{\tau^{\sigma(1+\alpha)}} d\tau.$$

The properties  $\sigma < \frac{1}{1+\alpha}$  and  $p > p_0$  together with  $\beta(p_0) = 1$  ensure the existence of the integral. For a technical reason, we use a less sharp estimate

$$K_{j+1} \leq K_0 + 2N_{1,T}N_2BT_0^{1-\beta(r)}K_j^{1+\alpha}.$$

Assume for a moment, that the inequality

$$(4.12) \quad T_0^{1-\beta(r)}K_0^\alpha < \left(\frac{\alpha}{1+\alpha}\right)^\alpha \frac{1}{2(1+\alpha)N_{1,T}N_2B}$$

is valid. Define  $K := K(T_0) := \frac{1+\alpha}{\alpha}K_0(T_0)$ . Then, by an elementary calculation, we find that

$$(4.13) \quad K_j < K \quad (j \geq 0),$$

$$(4.14) \quad 2N_{1,T}N_2BT_0^{1-\beta(r)}K^\alpha < \frac{1}{1+\alpha},$$

and

$$(4.15) \quad K \rightarrow 0 \quad \text{as} \quad K_0 \rightarrow 0.$$

We thus have an a priori bound for  $K_j$  under the condition (4.12).

In the following, we deduce conditions for  $T_0$  and  $a$  that guarantee (4.12). We start by noting that

$$(4.16) \quad t^\sigma \|e^{-tA_r}a\|_{U^p} \rightarrow 0 \quad \text{as} \quad t \rightarrow 0;$$

here the property  $p > r$  and the density of  $C_c(\Omega; \mathbb{C}^l)$  in  $L^r(\Omega, \mu; \mathbb{C}^l)$  are essential. Particularly, this proves that

$$(4.17) \quad K_0(T_0) \rightarrow 0 \quad \text{as} \quad T_0 \rightarrow 0.$$

Next, we verify that the assumptions of the theorem imply the validity of (4.12).

If  $r > p'_0$  (consequently  $\beta(r) < 1$ ), the condition  $T_0 < C\|a\|_{U^r}^{-\frac{\alpha}{1-\beta(r)}}$  ensures the validity of (4.12) since (A) implies that  $K_0/\|a\|_{U^r}$  is bounded by  $M_T$ .

If  $p_0 > p^\circ$  and  $r = p_0$ , the convergence (4.17) shows that (4.12) is valid, whenever  $T_0$  is small. Moreover, since  $\beta(p_0) = 1$ , (4.12) does not include  $T_0$  explicitly, so that (4.12) is satisfied with  $T_0 = T$  whenever

$$\|a\|_{U^{p_0}} < \frac{\alpha}{1+\alpha} \left( \frac{1}{2(1+\alpha)BN_{1,T}N_2M_T^\alpha} \right)^{\frac{1}{\alpha}}.$$

Furthermore, if  $M_T$  and  $N_{1,T}$  are uniformly bounded in  $0 < T < \infty$ ,  $K$  is finite for  $T_0 = \infty$  and (4.12) is satisfied for small  $\|a\|_{U^{p_0}}$  as well. We thus see that (4.12) holds under all assumptions of the theorem.

As the rest of the proof literally follows the proof of Giga [15, p. 195], we omit the details.  $\square$

**Remark 4.5.** A detailed version of the preceding proof can also be found in the PhD-thesis of the author [34, Thm. 6.3.5].

**Proof of Theorem 1.3.** To prove the existence of  $u$  apply Theorem 4.4 with  $p_- := 3/2 - \varepsilon$ ,  $p_+ := 3 + \varepsilon$ , and  $p^\circ$  any number between  $p_-$  and  $p_+$ . Let  $U^p := L^p_\sigma(\Omega)$  and  $P_p := \mathbb{P}_p$ . Define  $\Gamma := \operatorname{div} : W_0^{1,p}(\Omega; \mathbb{C}^{3 \times 3}) \rightarrow L^p(\Omega; \mathbb{C}^3)$  and  $Gu := u \otimes u$ . This gives  $\alpha = 1$ . The estimate in (A) is given by Theorem 1.2 and the estimate in (N) follows by dualizing the gradient estimates in Theorem 1.2 and by performing the calculation in Remark 4.2.

To obtain the control on the gradient of  $u$  in the case  $a \in L^3_\sigma(\Omega)$ , recall the successive approximation defined in (4.1) and (4.2)

$$u_{j+1} = u_0 + Su_j,$$

but write the nonlinearity of the Navier-Stokes equations as  $\mathbb{P}(u_j \cdot \nabla)u_j$ . Let  $3 \leq q < 3 + \varepsilon$  and let  $p$  be subject to (4.9). Note that in (4.9)  $r$  is now equal to 3. Choosing  $p^\circ := 3$  and  $p_-$ ,  $p_+$ , and  $\alpha$  as above, we find that for any  $3 < p < \min\{3 + \varepsilon, 6\}$  the condition in (4.9) is satisfied. Choose additionally  $p$  and  $w$  such that

$$\frac{1}{p} > \frac{1}{3} - \frac{1}{q} \quad \text{and} \quad \frac{1}{w} = \frac{1}{q} + \frac{1}{p}.$$

The conditions on  $p$  and  $q$  already imply  $3/2 < w < 3$ . Finally, define for  $T_0 > 0$

$$K'_j := \sup_{0 < s < T_0} s^{1-\frac{3}{2q}} \|\nabla u_j(s)\|_{L^q(\Omega; \mathbb{C}^9)} \quad (j \in \mathbb{N}_0).$$

By Theorem 1.2 we derive

$$\|\nabla u_{j+1}(t)\|_{L^q(\Omega; \mathbb{C}^9)} \leq CK'_j K \int_0^t (t-s)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{w}-\frac{1}{q})} s^{\frac{3}{2q}-1-\frac{3}{2}(\frac{1}{3}-\frac{1}{p})} ds,$$

where  $K$  is given above (4.13) and where (4.13) was used. Since  $1/w - 1/q = 1/p$ , we find

$$\int_0^t (t-s)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{w}-\frac{1}{q})} s^{\frac{3}{2q}-1-\frac{3}{2}(\frac{1}{3}-\frac{1}{p})} ds = t^{\frac{3}{2q}-1} \int_0^1 (1-x)^{-\frac{1}{2}-\frac{3}{2p}} x^{\frac{3}{2q}-1-\frac{3}{2}(\frac{1}{3}-\frac{1}{p})} dx.$$

This proves that

$$K'_{j+1} \leq K'_0 + CKK'_j.$$

Since  $K$  converges to zero as  $T_0 \rightarrow 0$  by (4.15) and since  $K \leq C\|a\|_{L^3_\sigma(\Omega)}$  by definition of  $K$ , one can choose  $CK \leq 1/2$  by either taking  $T_0$  or  $\|a\|_{L^3_\sigma(\Omega)}$  small. Inductively, one concludes

$$K'_j < 2K'_0 \quad (j \in \mathbb{N}),$$

which gives the control on the gradient of  $u$  in all considered cases by letting  $j \rightarrow \infty$ .  $\square$

**Remark 4.6.** To see that (N) is indeed easier to verify than (2') consult [34, Thm. 5.2.22]. This theorem establishes all estimates that are required to employ Theorem 4.4 but it only uses the analyticity of the Stokes semigroup on  $L^p_\sigma(\Omega)$  for  $3/2 - \varepsilon < p < 3 + \varepsilon$ , an  $L^2$ -theory of the fractional powers of the Stokes operator [23, Cor. 5.2], Sobolev's inequality, and complex interpolation.

**4.2. An approach via maximal  $L^q$ -regularity.** Recall that  $\mathbb{E}$  was given as the maximal regularity space

$$\mathbb{E} = W^{1,q}(0, \infty; L^p_\sigma(\Omega)) \cap L^q(0, \infty; \mathcal{D}(A_p)).$$

We say that functions  $u : (0, \infty) \rightarrow L^p_\sigma(\Omega)$  and  $\pi : (0, \infty) \rightarrow L^p(\Omega)$  are strong solutions to (NSE) if  $u \in \mathbb{E}$ ,  $u$  attains the initial condition in the sense of traces, and if

$$\int_\Omega \partial_t u(t) \cdot \overline{w} \, dx + \int_\Omega \nabla u(t) \cdot \overline{\nabla w} \, dx + \int_\Omega (u(t) \cdot \nabla) u(t) \cdot \overline{w} \, dx - \int_\Omega \pi \overline{\operatorname{div}(w)} \, dx = \int_\Omega f \cdot \overline{w} \, dx$$

holds for every  $w \in W_0^{1,p'}(\Omega; \mathbb{C}^3)$  and almost every  $t > 0$ . Here  $p'$  denotes the Hölder conjugate exponent of  $p$ .

In order to derive the existence of solutions to the Navier–Stokes equations via maximal  $L^q$ -regularity one usually performs the following steps:

- (1) Recast the Navier–Stokes equations on the subspace  $L^p_\sigma(\Omega)$  as

$$(\text{PNSE}) \quad \begin{cases} \partial_t u + A_p u = \mathbb{P}_p f - \mathbb{P}_p(u \cdot \nabla)u \\ u(0) = a. \end{cases}$$

- (2) Replace the term  $\mathbb{P}_p(u \cdot \nabla)u$  by  $\mathbb{P}_p(v \cdot \nabla)v$ , with  $v \in \mathbb{E}$ , and show that for all  $v \in \mathbb{E}$  we have  $\mathbb{P}_p(v \cdot \nabla)v \in L^q(0, \infty; L^p_\sigma(\Omega))$ . For fixed  $v \in \mathbb{E}$  the maximal  $L^q$ -regularity of  $A_p$  provides then a unique solution  $u_v \in \mathbb{E}$  to the corresponding linear problem.
- (3) For  $f$  and  $a$  fixed, show that the linear operator mapping  $v$  to  $u_v$  has a fixed point.

To verify (2) it is essential to have good embeddings of  $\mathbb{E}$  into a space of the form

$$L^r(0, \infty; W^{1,p}(\Omega; \mathbb{C}^3)),$$

for some suitable  $r > q$ , which is desired to be as large as possible. To obtain this embedding, the following two results are of great importance. The first result deals with embedding properties of the domain of the Stokes operator and is in the case  $p = 2$  due to Brown and Shen [3, Thm. 2.12], see also Mitrea and Monniaux [23, Thm. 5.3], and in the case  $p > 2$  due to Mitrea and Wright [24, Thm. 10.15].

**Theorem 4.7.** *Let  $\Omega \subset \mathbb{R}^3$  be a bounded Lipschitz domain.*

- (1) *In the case  $p = 2$  the continuous embedding  $\mathcal{D}(A_2) \subset H^{3/2,2}(\Omega; \mathbb{C}^3)$  is valid.*
- (2) *There exists  $\delta \in (0, 1]$  such that for all  $2 < p < 3 + \delta/(1 + \delta)$  and all  $s \in [0, 1)$  the continuous embedding*

$$\mathcal{D}(A_p) \subset H^{s+1/p,p}(\Omega; \mathbb{C}^3)$$

*holds.*

The second result that is needed is a mixed derivative theorem. The version presented here is essentially due to Denk and Kaip [6, Lem. 2.61] and is extended here to non-vanishing functions at  $t = 0$  by a reflection argument.

**Theorem 4.8.** *Let  $\Omega \subset \mathbb{R}^d$  be a bounded Lipschitz domain and  $1 < p, q < \infty$ . Then, for every  $s \geq 0$  and  $\sigma \in [0, 1]$  the continuous embedding*

$$W^{1,q}(0, \infty; L^p(\Omega)) \cap L^q(0, \infty; H^{s,p}(\Omega)) \subset H^{\sigma,q}(0, \infty; H^{(1-\sigma)s,p}(\Omega))$$

*holds.*

*Proof.* By [6, Lem. 2.61] there is a continuous embedding

$$W_0^{1,q}(0, \infty; L^p(\mathbb{R}^d)) \cap L^q(0, \infty; H^{s,p}(\mathbb{R}^d)) \subset H^{\sigma,q}(0, \infty; H^{(1-\sigma)s,p}(\mathbb{R}^d)).$$

To obtain the same embedding with  $\mathbb{R}^d$  replaced by  $\Omega$ , extend functions on  $\Omega$  to all of  $\mathbb{R}^d$  by employing Stein's extension operator [31, Thm. VI.3.5]. Next, for general

$$u \in W^{1,q}(0, \infty; L^p(\Omega)) \cap L^q(0, \infty; H^{s,p}(\Omega)),$$

extend  $u$  to a function  $\tilde{u}$  on  $\mathbb{R}$  by an even reflection and multiply the extended function by a smooth cut-off function  $\varphi$  that is one on  $[0, \infty)$  and zero on  $(-\infty, -1]$ . Finally, employ the embedding for vanishing trace at  $t = 0$  to the shifted function  $[\tilde{u}\varphi](\cdot - 2)$ .  $\square$

**Proposition 4.9.** *Let  $\Omega \subset \mathbb{R}^3$  be a bounded Lipschitz domain,  $\delta > 0$  as in Theorem 4.7, and  $2 \leq p < 3 + \delta/(1 + \delta)$ . Then, for*

$$\begin{cases} 1 < s < 1 + \frac{1}{p}, & \text{if } p > 2 \\ 1 < s \leq \frac{3}{2}, & \text{if } p = 2 \end{cases}$$

*and  $1 < q < s/(s - 1)$  the continuous embedding*

$$\mathbb{E} \subset L^{\frac{sq}{s-sq+q}}(0, \infty; W^{1,p}(\Omega; \mathbb{C}^3))$$

*holds.*

*Proof.* The proposition readily follows by combining Theorems 4.7 and 4.8 together with Sobolev's embedding theorem.  $\square$

Having a suitable embedding of  $\mathbb{E}$  at hand, we can start to estimate the nonlinear term. To do so, the following theorem of Brown and Shen [3, Thm. 3.1] is the final ingredient.

**Theorem 4.10.** *Let  $\Omega \subset \mathbb{R}^3$  be a bounded Lipschitz domain. Then there exists a constant  $C > 0$  depending only on the Lipschitz character of  $\Omega$  such that*

$$\|u\|_{L^\infty(\Omega; \mathbb{C}^3)} \leq C \|\nabla u\|_{L^2(\Omega; \mathbb{C}^9)}^{1/2} \|A_2 u\|_{L^2_\sigma(\Omega)}^{1/2} \quad (u \in \mathcal{D}(A_2)).$$

The following lemma gives the estimate of the nonlinear term and thereby concludes Step (2) of our three steps agenda.

**Lemma 4.11.** *Let  $\Omega \subset \mathbb{R}^3$  be a bounded Lipschitz domain,  $\delta > 0$  be as in Theorem 4.7, and  $2 \leq p < 3 + \delta/(\delta + 1)$ . In the case  $p = 2$ , let  $2 \leq q < \infty$ , and in the case  $p > 2$ , let*

$$\frac{2(p+1)}{3} < q < \infty.$$

*Then, there exists  $C > 0$  such that for all  $v, w \in \mathbb{E}$*

$$\|(v \cdot \nabla)w\|_{L^q(0, \infty; L^p(\Omega; \mathbb{C}^3))} \leq C \|v\|_{\mathbb{E}} \|w\|_{\mathbb{E}}.$$

*Proof.* Let  $v, w \in \mathbb{E}$ . Since  $p \geq 2$  we find  $\mathcal{D}(A_p) \subset \mathcal{D}(A_2)$  so that  $v(t) \in \mathcal{D}(A_2)$  for almost every  $t > 0$ . By Theorem 4.10 there exists a constant  $C > 0$  such that

$$\|v(t)\|_{L^\infty(\Omega; \mathbb{C}^3)} \leq C \|\nabla v(t)\|_{L^2(\Omega; \mathbb{C}^9)}^{1/2} \|A_2 v(t)\|_{L^2_\sigma(\Omega)}^{1/2} \quad (\text{a.e. } t > 0).$$

Thus, there exists a constant  $C > 0$  such that

$$\int_0^\infty \|(v(t) \cdot \nabla)w(t)\|_{L^p(\Omega; \mathbb{C}^3)}^q dt \leq C \int_0^\infty \|\nabla v(t)\|_{L^2(\Omega; \mathbb{C}^9)}^{q/2} \|A_2 v(t)\|_{L_\sigma^2(\Omega)}^{q/2} \|\nabla w(t)\|_{L^p(\Omega; \mathbb{C}^9)}^q dt.$$

An application of Hölder's inequality in space and time shows with a different constant  $C$

$$\begin{aligned} &\leq C \left( \int_0^\infty \|\nabla v(t)\|_{L^p(\Omega; \mathbb{C}^9)}^{3q} dt \right)^{\frac{1}{6}} \left( \int_0^\infty \|\nabla w(t)\|_{L^p(\Omega; \mathbb{C}^9)}^{3q} dt \right)^{\frac{1}{3}} \\ &\quad \cdot \left( \int_0^\infty \|A_p v(t)\|_{L_\sigma^p(\Omega)}^q dt \right)^{\frac{1}{2}}. \end{aligned}$$

Finally, we would like to appeal to Proposition 4.9. For this purpose, we have to ensure that there exists a number  $s$  subject to the premises given in the very same proposition, such that

$$1 < q < \frac{s}{s-1} \quad \text{and} \quad 3q = \frac{sq}{s-sq+q}.$$

One readily verifies that  $s$  given by

$$s := \frac{3q}{3q-2}$$

meets these requirements. Now, we can use Proposition 4.9 to estimate

$$\left( \int_0^\infty \|\nabla v(t)\|_{L^p(\Omega; \mathbb{C}^9)}^{3q} dt \right)^{\frac{1}{6}} \left( \int_0^\infty \|\nabla w(t)\|_{L^p(\Omega; \mathbb{C}^9)}^{3q} dt \right)^{\frac{1}{3}} \leq C \|v\|_{\mathbb{E}}^{q/2} \|w\|_{\mathbb{E}}^q.$$

This concludes the proof.  $\square$

Next, let  $p$  and  $q$  be as in Lemma 4.11. Moreover, fix  $f \in L^q(0, \infty; L^p(\Omega; \mathbb{C}^3))$  and  $a \in (L_\sigma^p(\Omega); \mathcal{D}(A_p))_{1-1/q, q}$ . Let

$$\mathcal{T}_{(f,a)} : \mathbb{E} \rightarrow \mathbb{E}$$

be the mapping that maps  $v$  to  $u_v$ , where  $u_v$  is given by

$$\begin{cases} \partial_t u_v + A_p u_v = \mathbb{P}_p f - \mathbb{P}_p(v \cdot \nabla)v & t > 0 \\ u_v(0) = a. \end{cases}$$

By Lemma 4.11 the right-hand side lies in  $L^q(0, \infty; L_\sigma^p(\Omega))$ , so that the maximal  $L^q$ -regularity of  $A_p$ , see Theorem 2.6, implies that the solution  $u_v$  indeed exists and that  $u_v \in \mathbb{E}$ . It follows that for each  $f$  and  $a$  chosen in the spaces above, the mapping  $\mathcal{T}_{(f,a)}$  is well-defined. The following theorem shows that  $\mathcal{T}_{(f,a)}$  has a unique fixed point, provided  $f$  and  $a$  are small enough. The proof follows the lines of Saal [26, Thm. 1.2].

**Theorem 4.12.** *Let  $\Omega \subset \mathbb{R}^3$  be a bounded Lipschitz domain,  $\varepsilon > 0$  be the minimal  $\varepsilon$  of Theorems 2.1 and 2.6,  $\delta > 0$  as in Theorem 4.7, and  $2 \leq p < 3 + \min\{\varepsilon, \delta/(\delta+1)\}$ . In the case  $p = 2$ , let  $2 \leq q < \infty$ , and in the case  $p > 2$ , let*

$$\frac{2(p+1)}{3} < q < \infty.$$

*Then, there exists a constant  $C > 0$ , such that for all*

$$(f, a) \in L^q(0, \infty; L^p(\Omega; \mathbb{C}^3)) \times (L_\sigma^p(\Omega), \mathcal{D}(A_p))_{1-1/q, q} =: \mathbb{F}$$

*with*

$$\|(f, a)\|_{\mathbb{F}} < C,$$

*there exists a unique fixed point  $u \in \mathbb{E}$  of  $\mathcal{T}_{(f,a)}$ .*



*Proof.* Let  $\tau_0 : \mathbb{E} \rightarrow (L_\sigma^p(\Omega), \mathcal{D}(A_p))_{1-1/q, q}$  be the trace operator at zero  $t = 0$ . This operator is well-defined due to the trace method for real interpolation, see, e.g., [22, Sec. 1]. Define

$$\mathbb{F}_\sigma := L^q(0, \infty; L_\sigma^p(\Omega)) \times (L_\sigma^p(\Omega), \mathcal{D}(A_p))_{1-1/q, q}$$

and the space of solutions corresponding to zero initial data

$$\mathbb{E}_0 := \tau_0^{-1}(\{0\}).$$

Furthermore, let  $L^{-1} : \mathbb{F}_\sigma \rightarrow \mathbb{E}$  denote the solution operator to the problem

$$\begin{cases} \partial_t u + A_p u = g \\ u(0) = a, \end{cases}$$

which is well-defined due to the maximal  $L^q$ -regularity of  $A_p$  stated in Theorem 2.6. Fix  $(f, a) \in \mathbb{F}$  and reformulate the operator  $\mathcal{T}_{(f, a)}$  by means of  $L^{-1}$  as

$$\mathcal{T}_{(f, a)}(v) = L^{-1}(\mathbb{P}_p f - \mathbb{P}_p(v \cdot \nabla)v, a).$$

Define  $\tilde{v} := L^{-1}(\mathbb{P}_p f, a)$  and conclude that by the linearity of  $L^{-1}$ ,

$$\mathcal{T}_{(f, a)}(v) = L^{-1}(-\mathbb{P}_p(v \cdot \nabla)v, 0) + \tilde{v}.$$

Next, let  $\bar{v} := v - \tilde{v}$ . By rearranging we see that  $\mathcal{T}_{(f, a)}$  has a fixed point if and only if

$$\bar{v} = L^{-1}(-\mathbb{P}_p(\bar{v} + \tilde{v}) \cdot \nabla)(\bar{v} + \tilde{v}), 0).$$

Thus,  $\mathcal{T}_{(f, a)}$  has a fixed point if and only if the mapping

$$\mathcal{S}_{(f, a)} : \mathbb{E}_0 \rightarrow \mathbb{E}_0, \quad \bar{v} \mapsto L^{-1}(-\mathbb{P}_p(\bar{v} + \tilde{v}) \cdot \nabla)(\bar{v} + \tilde{v}), 0)$$

has a fixed point. We proceed by showing that the contraction mapping principle is applicable to the restriction of  $\mathcal{S}_{(f, a)}$  to a ball  $B(0, r) \subset \mathbb{E}_0$  with a suitable radius.

Let  $r > 0$  to be chosen and  $\bar{v} \in B(0, r)$ . Then, by using (2.3), the boundedness of the Helmholtz projection on  $L^p(\Omega; \mathbb{C}^3)$  as well as Lemma 4.11, we find

$$\begin{aligned} \|\mathcal{S}_{(f, a)}(\bar{v})\|_{\mathbb{E}_0} &\leq C\|\mathbb{P}_p(\bar{v} + \tilde{v}) \cdot \nabla)(\bar{v} + \tilde{v})\|_{L^q(0, \infty; L_\sigma^p(\Omega))} \\ &\leq C\{\|\bar{v}\|_{\mathbb{E}_0}^2 + \|\bar{v}\|_{\mathbb{E}_0}\|(f, a)\|_{\mathbb{F}} + \|(f, a)\|_{\mathbb{F}}^2\} \\ &\leq C\{r^2 + r\|(f, a)\|_{\mathbb{F}} + \|(f, a)\|_{\mathbb{F}}^2\}. \end{aligned}$$

Taking  $\|(f, a)\|_{\mathbb{F}} < r$ , we obtain that

$$3Cr^2 < r \quad \Leftrightarrow \quad r < \frac{1}{3C}.$$

Consequently, for every  $r$  satisfying the preceding inequality and for all  $(f, a)$  satisfying the smallness condition in  $\mathbb{F}$  the calculation above shows that  $\mathcal{S}_{(f, a)}$  maps  $B(0, r)$  into itself. To show that the restriction of  $\mathcal{S}_{(f, a)}$  to a ball (of proper size) is a strict contraction, calculate with the same reasoning as above

$$\begin{aligned} \|\mathcal{S}_{(f, a)}(\bar{v}_1) - \mathcal{S}_{(f, a)}(\bar{v}_2)\|_{\mathbb{E}_0} &\leq C\{\|\mathbb{P}_p(\bar{v}_1 + \tilde{v}) \cdot \nabla)(\bar{v}_1 - \bar{v}_2)\|_{L^q(0, \infty; L_\sigma^p(\Omega))} \\ &\quad + \|\mathbb{P}_p(\bar{v}_1 - \bar{v}_2) \cdot \nabla)(\bar{v}_2 + \tilde{v})\|_{L^q(0, \infty; L_\sigma^p(\Omega))}\} \\ &\leq C\{\|\bar{v}_1\|_{\mathbb{E}_0} + \|(f, a)\|_{\mathbb{F}} + \|\bar{v}_2\|_{\mathbb{E}_0}\}\|\bar{v}_1 - \bar{v}_2\|_{\mathbb{E}_0}. \end{aligned}$$

Thus,  $\mathcal{S}_{(f, a)}$  is a strict contraction if  $r < 1/3C$ . Hence, the contraction mapping principle is applicable and proves the existence of a unique fixed point of  $\mathcal{S}_{(f, a)}$  in  $B(0, r)$ .  $\square$

**Proof of Theorem 1.4.** The existence of  $u$  follows already by Theorem 4.12. To retrieve the pressure, note that the pressure is the sum of the gradient fields originating from

$$\mathbb{P}_p(u(t) \cdot \nabla)u(t) = (u(t) \cdot \nabla)u(t) + (\mathbb{P}_p - \text{Id})(u(t) \cdot \nabla)u(t)$$

and

$$\mathbb{P}_p f(t) = f(t) + (\mathbb{P}_p - \text{Id})f(t),$$

and from the pressure function which comes from the characterization of the domain of the Stokes operator, cf. Theorem 2.2. Note that this function lies a priori only in  $L^2(\Omega)$  but since  $u \in W_{0,\sigma}^{1,p}(\Omega)$  and  $A_p u \in L_\sigma^p(\Omega)$  one finds that the pressure lies in  $L^p(\Omega)$  by using Bogovskii's operator. For details, see [34, Thm. 5.2.11]. The uniqueness of  $u$  follows by the uniqueness statement in Theorem 4.12.  $\square$

## REFERENCES

- [1] H. AMANN. *On the strong solvability of the Navier–Stokes equations*. J. Math. Fluid Mech. **2** (2000), no. 1, 16–98.
- [2] W. ARENDT, C. J. K. BATTY, M. HIEBER, and F. NEUBRANDER. *Vector-valued Laplace Transforms and Cauchy Problems*. Monographs in Mathematics, vol. 96, Birkhäuser, Basel-Boston-Berlin, 2001.
- [3] R. M. BROWN and Z. SHEN. *Estimates for the Stokes operator in Lipschitz domains*. Indiana Univ. Math. J. **44** (1995), no. 4, 1183–1206.
- [4] P. CLÉMENT and S. LI. *Abstract parabolic quasilinear equations and application to a groundwater flow problem*. Adv. Math. Sci. Appl. **3** (1993/1994), Special Issue, 17–32.
- [5] R. DENK, M. HIEBER, and J. PRÜSS.  *$\mathcal{R}$ -boundedness, Fourier multipliers and problems of elliptic and parabolic type*. Mem. Amer. Math. Soc. **166** (2003), no. 788.
- [6] R. DENK and M. KAIP. *General Parabolic Mixed Order Systems in  $L_p$  and Applications*. Operator Theory: Advances and Applications, vol. 239, Birkhäuser/Springer, Cham, 2013.
- [7] P. DEURING. *The Stokes resolvent in 3D domains with conical boundary points: nonregularity in  $L^p$ -spaces*. Adv. Differential Equations **6** (2001), no. 2, 175–228.
- [8] P. DEURING and W. VON WAHL. *Strong solutions of the Navier–Stokes system in Lipschitz bounded domains*. Math. Nachr. **171** (1995), 111–148.
- [9] E. FABES, O. MENDEZ, and M. MITREA. *Boundary layers on Sobolev-Besov spaces and Poisson's equation for the Laplacian in Lipschitz domains*. J. Funct. Anal. **159** (1998), no. 2, 323–368.
- [10] H. FUJITA and T. KATO. *On the nonstationary Navier–Stokes system*. Rend. Sem. Mat. Univ. Padova **32** (1962), 243–260.
- [11] G. P. GALDI. *An introduction to the mathematical theory of the Navier-Stokes equations. Steady-state problems*. Springer Monographs in Mathematics. Springer, New York, 2011.
- [12] M. GEISSERT, M. HESS, M. HIEBER, C. SCHWARZ, and K. STAVRAKIDIS. *Maximal  $L^p - L^q$ -Estimates for the Stokes Equation: a Short Proof of Solonnikov's Theorem*. J. Math. Fluid Mech. **12** (2010), 47–60.
- [13] J. GENG and J. KILTY. *The  $L^p$  regularity problem for the Stokes system on Lipschitz domains*. J. Differential Equations **259** (2015), no. 4, 1275–1296.
- [14] Y. GIGA. *Domains of fractional powers of the Stokes operator in  $L_r$  spaces*. Arch. Ration. Mech. Anal. **89** (1985), no. 3, 251–265.
- [15] Y. GIGA. *Solutions for semilinear parabolic equations in  $L^p$  and regularity of weak solutions of the Navier–Stokes system*. J. Differential Equations **62** (1986), no. 2, 186–212.
- [16] Y. GIGA and H. SOHR. *Abstract  $L^p$  estimates for the Cauchy problem with applications to the Navier–Stokes equations in exterior domains*. J. Funct. Anal. **102** (1991), no. 1, 72–94.
- [17] M. HAASE. *The Functional Calculus for Sectorial Operators*. Operator Theory: Advances and Applications, vol. 169, Birkhäuser, Basel, 2006.
- [18] T. KATO. *Strong  $L^p$ -solutions of the Navier–Stokes equations in  $\mathbb{R}^m$ , with applications to weak solutions*. Math. Z. **187** (1984), no. 4, 471–480.
- [19] P. C. KUNSTMANN and L. WEIS. *Maximal  $L_p$ -regularity for parabolic equations, Fourier multiplier theorems and  $H^\infty$ -functional calculus*. In Functional analytic methods for evolution equations, Lecture Notes in Mathematics, vol. 1855, Springer, Berlin, 2004, 65–311.
- [20] P. C. KUNSTMANN and L. WEIS. *New criteria for the  $H^\infty$ -calculus and the Stokes operator on bounded Lipschitz domains*. J. Evol. Equ. (2016), 1–23.
- [21] O. A. LADYZHENSKAYA. *The mathematical theory of viscous incompressible flow*. Mathematics and its Applications, vol. 2, Gordon and Breach Science Publishers, New York-London-Paris, 1969.
- [22] A. LUNARDI. *Interpolation Theory*. Scuola Normale Superiore di Pisa (Nuova Serie), Pisa, 2009.

- [23] M. MITREA and S. MONNIAUX. *The regularity of the Stokes operator and the Fujita-Kato approach to the Navier-Stokes initial value problem in Lipschitz domains*. J. Funct. Anal. **254** (2008), no. 6, 1522–1574.
- [24] M. MITREA and M. WRIGHT. *Boundary value problems for the Stokes system in arbitrary Lipschitz domains*. Astérisque **344** (2012).
- [25] L. NIRENBERG. *On elliptic partial differential equations*. Ann. Scuola Norm. Sup. Pisa (3) **13** (1959), 115–162.
- [26] J. SAAL. *Strong solutions for the Navier–Stokes equations on bounded and unbounded domains with a moving boundary*. Electron. J. Differ. Equ. Conf. **15** (2007), 365–375.
- [27] Z. SHEN. *Bounds on Riesz transforms on  $L^p$ -spaces for second order elliptic operators*. Ann. Inst. Fourier (Grenoble) **55** (2005), no. 1, 173–197.
- [28] Z. SHEN. *Resolvent estimates in  $L^p$  for the Stokes operator in Lipschitz domains*. Arch. Ration. Mech. Anal. (2012), no. 2, 395–424.
- [29] H. SOHR. *The Navier-Stokes equations. An elementary functional analytic approach*. Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Verlag, Basel, 2001.
- [30] V. A. SOLONNIKOV. *Estimates for solutions of nonstationary Navier–Stokes equations*. J. Soviet Math. **8** (1977), no. 4, 467–529.
- [31] E. M. STEIN. *Singular integrals and differentiability properties of functions*. Princeton University Press, Princeton, 1986.
- [32] M. E. TAYLOR. *Incompressible fluid flows on rough domains*. In: Semigroups of operators: theory and applications (Newport Beach, CA, 1998). Progr. Nonlinear Differential Equations Appl., vol. 42, Birkhäuser, Basel, 2000, 320–334.
- [33] R. TEMAM. *Navier-Stokes equations and nonlinear functional analysis*. CBMS-NFS Regional Conference Series in Applied Mathematics, 66. SIAM, Philadelphia, PA, 1995.
- [34] P. TOLKSDORF. *On the  $L^p$ -theory of the Navier–Stokes equations on Lipschitz domains*. Technische Universität Darmstadt, Darmstadt, 2017, <http://tuprints.ulb.tu-darmstadt.de/5960/>.

FACHBEREICH MATHEMATIK, TECHNISCHE UNIVERSITÄT DARMSTADT, SCHLOSSGARTENSTR. 7, 64289 DARMSTADT, GERMANY

*E-mail address:* `tolksdorf@mathematik.tu-darmstadt.de`